AD 607691

~~704013~~

Technical Memorandum

ACCURACY OF ORBIT DETERMINATION

by
J. V. Breakwell

LMSD-704013 October 1960

12

Work done under Air Force Contract No. AF 04(647)-595

LOCKHEED ~~MISSILES AND SPACE DIVISION~~
SUNNYVALE, CALIFORNIA

ACCURACY OF ORBIT DETERMINATION

by

J. V. Breakwell

INTRODUCTION

This report describes a computational program, now in existence, for estimating the accuracy with which satellite positions can be calculated at any time on the basis of observations at various stations. The error in computed position is due to two main causes, at least if station location uncertainties are ignored. These are: (A) observation errors, (B) fluctuations in the orbital decay rate due to drag. It is assumed here that uncertainties in the earth's gravitational field have negligible effect on the satellite position.

The treatment of (A) follows standard statistical practice: Orbital parameters are adjusted to a least-squares fit to the observations. The covariances of the estimates of the orbit parameters are then obtainable quite easily from the variances of the observation errors, assuming that the different measurements are statistically independent with zero bias.

The orbital parameters are seven in number, rather than six: The (mean) orbital decay rate is a seventh parameter (but numbered 5 in § 2) to be deduced from the observations rather than from prior knowledge of the atmosphere. In the case of a received Doppler signal, the emitting frequency, assumed constant during a single "pass," (i.e., during the reception of the signal at a single station on a single revolution) constitutes an additional parameter per Doppler pass. These additional parameters, however, are easily eliminated from the calculation, as in reference [2].

The treatment of (B) is as follows: Fluctuations in the deceleration due to drag are considered to constitute a stationary time series with exponentially decreasing auto-correlation. The effect of these drag fluctuations on altitude is ignored: Only the angular position along the orbit is considered to be affected. A typical correlation time for the drag fluctuations might be 3 hours. This would correspond in § 6 to $\delta_c \cong 4\pi$. The r.m.s. value of the drag fluctuations may be assumed to be a small fraction, e.g., one fifteenth, of the drag at perigee. A 3-σ fluctuation of one fifth in density is not unreasonable over short intervals (i.e., in a single day). This is not to be confused with significant long-term changes in drag due to a gradual change in the relative positions of perigee and the sun.

§ 1. LEAST SQUARES PROCEDURE

The "maximum likelihood" estimation of orbital parameters from a set of observations $\tilde{q}_\ell$ with independent errors with various standard deviations σ_ℓ requires the minimization of the sum of the weighted squared residuals:

$$S = \sum_\ell \left(\frac{q(\lambda, t_\ell) - \tilde{q}_\ell}{\sigma_\ell} \right)^2 \tag{1.1}$$

when $q(\lambda, t_\ell)$ is the theoretical value for the ℓ-th observation, at time t_ℓ , based on a set λ of orbital parameters λ_α . The index ℓ is understood to run, say chronologically, over all observations, including simultaneous measurements of different physical quantities such as range, azimuth and elevation, in which case three (successive) t_ℓ's would be identical. The σ_ℓ's for all range measurements from similar radar equipment will be equal, say σ_S ; the σ_ℓ's for elevation measurements will all be equal, say σ_E ; etc.

By linearizing the $q(\lambda, t_\ell)$'s as functions of λ_α in the neighborhood of certain approximate values λ_α^o :

$$(1.2) \qquad q(\lambda, t_\ell) \cong q(\lambda^o, t_\ell) + \sum_\alpha \frac{\partial q(\lambda^o, t_\ell)}{\partial \lambda_\alpha^o} (\lambda_\alpha - \lambda_\alpha^o) \ ,$$

the estimation reduces to solution of the following set of simultaneous linear equations:

$$(1.3) \qquad \sum_\beta M_{\alpha\beta} (\lambda_\beta - \lambda_\beta^o) = V_\alpha \ ,$$

where

$$(1.4) \qquad \begin{cases} M_{\alpha\beta} = \sum_\ell \dfrac{1}{\sigma_\ell^2} \dfrac{\partial q(\lambda^o, t_\ell)}{\partial \lambda_\alpha^o} \dfrac{\partial q(\lambda^o, t_\ell)}{\partial \lambda_\beta^o} \\[2ex] V_\alpha = \sum_\ell \dfrac{1}{\sigma_\ell^2} \dfrac{\partial q(\lambda^o, t_\ell)}{\partial \lambda_\alpha^o} \left[\bar{q}_\ell - q_\ell(\lambda^o, t_\ell)\right] . \end{cases}$$

What is of interest to us here is the statistical theorem that the variances and covariances of our estimates λ_α of the parameters λ_α are just the elements of the inverse M^{-1} of the "information matrix" $M_{\alpha\beta}$.

It follows that if $\hat{x}_j = x_j(\hat{\lambda}, t^*)$ is a position coordinate x_j at some time t^* , computed from our estimated orbital parameters $\hat{\lambda}_\alpha$, then:

$$(1.5) \qquad \sigma_{\hat{x}_j}(t^*) = \sqrt{\sum_\alpha \sum_\beta (M^{-1})_{\alpha\beta} \frac{\partial x_j(\lambda, t^*)}{\partial \lambda_\alpha} \frac{\partial x_j(\lambda, t^*)}{\partial \lambda_\beta}}$$

§ 2. EVALUATION OF THE $\frac{\partial x}{\partial \lambda_\gamma}$'s

We proceed next to the forms for $\frac{\partial x_i}{\partial \lambda_\gamma}$ appropriate to a particular parametrization of the orbit. The quantities $\frac{\partial q(\lambda^o, t_i)}{\partial \lambda^o_\alpha}$ will later be related to the $\frac{\partial x_i}{\partial \lambda_\alpha}$'s at time t_i by some rather obvious trigonometry, and $M_{\alpha\beta}$ will be thus obtainable.

We describe an orbit, including its decay due to drag, by seven parameters:

$$(2.1)\quad \begin{cases} \lambda_1 = \dfrac{a}{a_N} - 1\ , \quad \lambda_2 = \epsilon \cos \beta\ , \quad \lambda_3 = \epsilon \sin \beta\ , \quad \lambda_4 = \dfrac{\mu^{1/2}}{a_N^{3/2}}\, t_\Omega\ , \\ \lambda_5 = -\dfrac{1}{a_N}\dfrac{da}{d\theta}\ , \quad \lambda_6 = -\,\Omega \sin i_N\ , \quad \lambda_7 = i\ . \end{cases}$$

Here a is the initial semi-major axis, a_N an a-priori "nominal" value for a ; ϵ is the eccentricity, β the "argument of perigee," t_Ω the time at the first ascending node; μ is the universal gravitational constant times the mass of the earth; $-\frac{da}{d\theta}$ is the rate of decay of the semi-major axis per angular distance θ from the first ascending node; Ω is the right-ascension of the ascending node, i the orbital inclination (i.e., the angle between the orbital plane and the equatorial plane taken as acute if and only if the orbit is Eastward); i_N is a nominal orbital inclination.

The earth's oblateness, of course, causes slow changes in the parameters Ω and β . The amounts of their changes, however, are known functions of the other orbital parameters as well as of the earth's oblateness coefficient J , which is known with reasonable accuracy. The accuracy

problem is therefore not appreciably worsened by the earth's oblateness which we shall therefore ignore.

At each point of a nominal orbit a right-handed coordinate system is used: x = horizontal in the nominal orbit plane, forward; y = (horizontal) perpendicular to the nominal orbit plane, to the left; z = vertically upward. It is not difficult to see that if $\Delta\lambda_\alpha$ denotes the difference $\lambda_\alpha - (\lambda_\alpha)_N$ between actual and nominal parameter values, and Δy the deviation perpendicular to the nominal plane, then (neglecting second-order differences);

$$\frac{\Delta y}{r} = \cos\theta\,\Delta\lambda_8 + \sin\theta\,\Delta\lambda_7$$

r being the local (radial) distance from the earth's center. Assuming that the eccentricity ϵ is small, we may replace this by:

$$\frac{\Delta y}{a_N} \cong \cos\theta\,\Delta\lambda_8 + \sin\theta\,\Delta\lambda_7 \tag{2.2}$$

Next, the radial distance r is known as a function of θ :

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta-\beta)} \cong a(1 - \lambda_2\cos\theta - \lambda_3\sin\theta)$$

$$\cong a_N(1 - \lambda_1 - \lambda_2\cos\theta - \lambda_3\sin\theta) ,$$

so that

$$\frac{\Delta z}{a_N} = \Delta\lambda_1 - \cos\theta\,\Delta\lambda_2 - \sin\theta\,\Delta\lambda_3 , \tag{2.3}$$

ignoring for the moment the effect of drag.

Next the time t at position θ differs from the first time t_p at perigee $(\theta = \beta)$ by:

$$t - t_p = \frac{a^{3/2}}{\mu^{1/2}} (E - \epsilon \sin E)$$

where E, the eccentric anomaly, is related to the true anomaly $(\theta - \beta)$ by:

$$\tan \frac{E}{2} = \sqrt{\frac{1-\epsilon}{1+\epsilon}} \tan \frac{\theta - \beta}{2}$$

Ignoring ϵ^2, we may write $E = \theta - \beta - \epsilon \sin(\theta - \beta)$ and

$$t - t_p = \frac{a^{3/2}}{\mu^{1/2}} \left[\theta - \beta - 2\epsilon \sin(\theta - \beta) \right] .$$

If t_Ω denotes the first time at the ascending node $(\theta = 0)$, it follows that:

$$t - t_\Omega = \frac{a^{3/2}}{\mu^{1/2}} \left[\theta - 2\epsilon \sin(\theta - \beta) - 2\epsilon \sin \beta \right]$$

$$= \frac{a^{3/2}}{\mu^{1/2}} \left[\theta - 2\lambda_2 \sin \theta - 2\lambda_3 (1 - \cos \theta) \right] .$$

Replacing a by $a_N(1 + \lambda_1)$ and ignoring terms quadratic in the λ's:

$$t - t_\Omega \cong \frac{a_N^{3/2}}{\mu^{1/2}} \left[\left(1 + \frac{3}{2}\lambda_1\right)\theta - 2\lambda_2 \sin \theta - 2\lambda_3 (1 - \cos \theta) \right], \tag{2.5}$$

so that the forward deviation Δx along the track at time t is given by:

(2.4) $$\frac{\Delta x}{a_N} \cong - \frac{\mu^{1/2}}{a_N^{3/2}} \Delta t = - \frac{3}{2} \theta \Delta\lambda_1 + 2 \sin \theta \Delta\lambda_2 + 2 (1 - \cos \theta) \Delta\lambda_3 - \Delta\lambda_4 .$$

To include drag, it has been shown elsewhere [1] that $\frac{\Delta e}{\Delta a} = \frac{I_1(z)}{I_0(z)}$ (cf. equation (3?) of [1]) where $z = \frac{ae}{H}$ and H is the atmospheric scale-height near perigee altitude. The contribution of D_5 to $\frac{z}{a_N}$ is thus (approximately)

$$-e\, D_5 \left[1 - \frac{I_1(z)}{I_0(z)} \cos (\theta - \theta_N) \right] .$$

The deviation in period, meanwhile, is $\Delta P = \frac{3}{2} P \frac{\Delta a}{a} = -\frac{3}{2} P a\, D_5$, and the accumulated time difference at position θ is $\int \Delta P\, dN$ where $N = \theta / 2\pi$. Hence the contribution of D_5 to $\frac{x}{a_N}$ is $+\frac{3}{4}\theta^2 D_5$.

Denoting the partial derivatives of $\frac{x}{a_N}$, $\frac{y}{a_N}$, $\frac{z}{a_N}$ w.r.t. [illegible] by x_i, y_i, z_i we now have (approximately):

$$
(\text{.5})\quad
\begin{cases}
x_1 = -\frac{3}{2}\theta \\
x_2 = 2 \sin \theta \\
x_3 = 2(1-\cos\theta) \\
x_4 = -1 \\
x_5 = \frac{3}{4}\theta^2 \\
x_6 = x_7 = 0
\end{cases}
\qquad
\begin{cases}
y_1 = y_2 = y_3 \\
\quad = y_4 = y_5 = 0 \\
y_6 = \cos\theta \\
y_7 = \sin\theta
\end{cases}
\qquad
\begin{cases}
z_1 = 1 \\
z_2 = -\cos\theta \\
z_3 = -\sin\theta \\
z_4 = 0 \\
z_5 = -e\left[1 - \frac{I_1(z_N)}{I_0(z_N)} \cos(\theta - \theta_N)\right] \\
z_6 = z_7 = 0 .
\end{cases}
$$

where z_N and θ_N [illegible]

§ 3. CALCULATION OF THE $\frac{\partial q}{\partial \lambda_\alpha}$'s

We come next to the calculation of the $\frac{\partial q}{\partial \lambda_\alpha}$'s. In the case of radar range and angle measurements the q's are functions of the path coordinates x , y , z , as well as of the position of the receiving station relative to the satellite's position. In the case of Doppler frequency measurement, the q is a function of both position and velocity along the path as well as of the relative position of the receiving station. We shall postpone the discussion of Doppler measurements until § 4.

It is clear that, for low altitude satellites, the partial derivatives $\frac{\partial q}{\partial \lambda_\alpha}$ vary much more rapidly with time during a pass than do the coordinate partial derivatives x_α , y_α , z_α themselves. We may thus get an approximate set of matrix elements $M_{\alpha\beta}$ arising from radar data by writing:

$$\frac{\partial q}{\partial \lambda_\alpha} = \frac{\partial q}{\partial x}\frac{\partial x}{\partial \lambda_\alpha} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial \lambda_\alpha} + \frac{\partial q}{\partial z}\frac{\partial z}{\partial \lambda_\alpha} = a_N\left(\frac{\partial q}{\partial x}x_\alpha + \frac{\partial q}{\partial y}y_\alpha + \frac{\partial q}{\partial z}z_\alpha\right) \tag{3.1}$$

$$= a_N \vec{\nabla} q \cdot \vec{r}_\alpha$$

and then replacing $\vec{r}_\alpha$ throughout a pass by its value at the middle of the pass. This is equivalent to smoothing all the radar data to yield only a single point per pass, and relying on three or more passes at sensibly different points along the orbit for the determination of all seven orbital parameters.

In the case of measurements of range S we have: $\vec{\nabla} q = \vec{\nabla} S = \hat{u}_1$, a unit vector along the line of sight. The contribution to $M_{\alpha\beta}$ of range measurements during a single pass, assuming that the measurements have

independent errors with r.m.s. value σ_S is thus:

$$M_{\alpha\beta}^{(S)} = \sum_{i} \left(\frac{a_N}{\sigma_S}\right)^2 \left(\hat{u}_1 \cdot \vec{r}_\alpha\right)\left(\vec{u}_1 \cdot \vec{r}_\beta\right) = \left(\frac{a_N}{\sigma_S}\right)^2 \vec{r}_\alpha \cdot Q_S \cdot \vec{r}_\beta \tag{3.2}$$

when Q_S is the matrix (dyadic) $\sum_i \hat{u}_1 \hat{u}_1$. Now if ϕ denotes the angular distance along the orbit, measured forward, from the point nearest the station, and if ρ denotes the minimum slant-range divided by the earth's radius R , and if α denotes the angular distance (at the center of the earth) between the station and the orbit plane, the components of the unit vector u_1 are approximately:

$$\ell_1 = \frac{\phi}{\sqrt{\rho^2+\varphi^2}} , \quad m_1 = \frac{\pm\alpha}{\sqrt{\rho^2+\varphi^2}} , \quad n_1 = \frac{\sqrt{\rho^2+\alpha^2}}{\sqrt{\rho^2+\varphi^2}} , \tag{3.3}$$

the + sign to be taken whenever the station lies to the right of the orbit (relative to the satellite's motion).

Next, the sum $\sum_i (\ )$ is replaced by $n\int_{-\varphi_m}^{\varphi_m} (\ )\, d\varphi$, where n is the number of independent radar measurements per geocentric radian of orbit, and φ_m is one half of the geocentric angular interval under radar observation from this station. Hence:

$$Q_S = \begin{Vmatrix} n\int_{-\varphi_m}^{\varphi_m} \frac{\varphi^2\, d\varphi}{\rho^2+\varphi^2} & 0 & 0 \\ 0 & n\int_{-\varphi_m}^{\varphi_m} \frac{\alpha^2\, d\varphi}{\rho^2+\varphi^2} & \pm n\int_{-\varphi_m}^{\varphi_m} \frac{\alpha\sqrt{\rho^2-\alpha^2}}{\rho^2+\varphi^2}\, d\varphi \\ 0 & \pm n\int_{-\varphi_m}^{\varphi_m} \frac{\alpha\sqrt{\rho^2-\alpha^2}}{\rho^2+\varphi^2}\, d\varphi & n\int_{-\varphi_m}^{\varphi_m} \frac{\rho^2-\alpha^2}{\rho^2+\varphi^2}\, d\varphi \end{Vmatrix} \tag{3.4}$$

where the off-diagonal terms $\Sigma \ell_1 m_1$ and $\Sigma \ell_1 n_1$ vanish since the interval of integration is symmetric about $\phi = 0$ and these integrands are odd functions of ϕ .

Evaluating the integrals:

$$(3.5) \qquad Q_S = \left\| \begin{array}{ccc} 2n\,(\phi_m - \rho\psi) & 0 & 0 \\ 0 & 2n\,\dfrac{\alpha^2}{\rho}\,\psi & \pm\, 2n\,\dfrac{\alpha}{\rho}\sqrt{\rho^2-\alpha^2}\,\psi \\ 0 & \pm\, 2n\,\dfrac{\alpha}{\rho}\sqrt{\rho^2-\alpha^2}\,\psi & 2n\left(\rho - \dfrac{\alpha^2}{\rho}\right)\psi \end{array} \right\|$$

where

$$(3.6) \qquad \psi = \mathrm{Tan}^{-1}\,\frac{\phi_m}{\rho} \;.$$

Turning next to the angle measurements we shall assume that the r.m.s. angular error in the vertical plane is equal to that in the plane through the line of sight perpendicular to the vertical plane, i.e., $(\cos E)\,\sigma_A = \sigma_E$, where E now denotes elevation angle and A azimuth. This assumption is not valid for very low elevations where σ_E is substantially greater than σ_A . However, these low elevation measurements are usually excluded; i.e., the value of ϕ_m corresponds to the minimum allowable E which is greater than zero.

Introducing two more unit vectors $\hat{u}_2$ and $\hat{u}_3$, which with $\hat{u}_1$ form a perpendicular triad, the contribution of angular errors to $M_{\alpha\beta}$ is

$$\sum_i \left(\frac{a_N}{\sigma_E}\right)^2 \frac{\left(\hat{u}_2 \cdot \vec{r}_\alpha\right)\left(\hat{u}_2 \cdot \vec{r}_\beta\right) + \left(\hat{u}_3 \cdot \vec{r}_\alpha\right)\left(\hat{u}_3 \cdot \vec{r}_\beta\right)}{S^2}$$

when S , the instantaneous slant-range, is approximately $R\sqrt{\rho^2+\varphi^2}$ thus the angle contribution to $M_{\alpha\beta}$ may be written:

$$(3.7) \qquad M_{\alpha\beta}^{(E)} = \left(\frac{a_N}{\sigma_E}\right)^2 \vec{r}_\alpha \cdot Q_E \cdot \vec{r}_\beta$$

where

$$Q_E = n\int_{-\varphi_m}^{\varphi_m} \frac{\left(\hat{u}_2\,\hat{u}_2 + \hat{u}_3\,\hat{u}_3\right)d\varphi}{\rho^2+\varphi^2}$$

Introducing the components ℓ_2 , m_2 , n_2 and ℓ_3 , m_3 , n_3 of $\hat{u}_2$ and $\hat{u}_3$ and making use of the identities $\ell_1^2+\ell_2^2+\ell_3^2=1$, etc., and $\ell_1 m_1+\ell_2 m_2+\ell_3 m_3=0$, etc., we obtain

$$(3.8) \quad Q_E = \left\| \begin{matrix} n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{1-\ell_1^2}{\rho^2+\varphi^2}\,d\varphi & -n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{\ell_1\,m_1}{\rho^2+\varphi^2}\,d\varphi & -n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{\ell_1\,n_1}{\rho^2+\varphi^2}\,d\varphi \\ -n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{\ell_1\,m_1}{\rho^2+\varphi^2}\,d\varphi & n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{1-m_1^2}{\rho^2+\varphi^2}\,d\varphi & -n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{m_1\,n_1}{\rho^2+\varphi^2}\,d\varphi \\ -n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{\ell_1\,n_1}{\rho^2+\varphi^2}\,d\varphi & -n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{m_1\,n_1}{\rho^2+\varphi^2}\,d\varphi & n\displaystyle\int_{-\varphi_m}^{\varphi_m}\frac{1-n_1^2}{\rho^2+\varphi^2}\,d\varphi \end{matrix} \right\|$$

$$= \left\| \begin{matrix} \frac{n}{\rho}\left(\psi+\frac{1}{2}\sin 2\psi\right) & 0 & 0 \\ 0 & \frac{2n\psi}{\rho}-\frac{n\alpha^2}{\rho^3}\left(\psi+\frac{1}{2}\sin 2\psi\right) & \mp\frac{n\alpha\sqrt{\rho^2-\alpha^2}}{\rho^3}\left(\psi+\frac{1}{2}\sin 2\psi\right) \\ 0 & \mp\frac{n\alpha\sqrt{\rho^2-\alpha^2}}{\rho^3}\left(\psi+\frac{1}{2}\sin 2\psi\right) & \frac{2n\psi}{\rho}-\frac{n(\rho^2-\alpha^2)}{\rho^3}\left(\psi+\frac{1}{2}\sin 2\psi\right) \end{matrix} \right\|$$

The total matrix $M_{\alpha\beta}$ from all radar observations is now:

$$(3.9) \qquad M_{\alpha\beta} = \sum_k \left[\left(\frac{a_N}{\sigma_S}\right)^2 \vec{r}_\alpha \cdot Q_S \cdot \vec{r}_\beta + \left(\frac{a_N}{R\sigma_E}\right)^2 \vec{r}_\alpha \cdot Q_E \cdot \vec{r}_\beta \right]$$

where the index k runs over the set of passes and where the parameters φ_m, α and ρ associated with each pass (each k) may be deduced from the three quantities: Local altitude h_k, maximum elevation E_k, and minimum allowable elevation E_o.

Thus

$$(3.10) \qquad \alpha = \frac{\pi}{2} - E_k - \sin^{-1}\left(\frac{R\cos E_k}{R+h_k}\right)$$

$$= \cos^{-1} \frac{R\cos^2 E_k + \sin E_k \sqrt{(R+h_k)^2 - R^2\cos^2 E_k}}{R+h_k},$$

$$(3.11) \qquad \rho \cong \sqrt{\left(\frac{h_k}{R}\right)^2 + \alpha^2},$$

$$\varphi_m = \cos^{-1}\frac{\cos\alpha'}{\cos\alpha},$$

where

$$\cos\alpha' = \frac{R\cos^2 E_o + \sin E_o \sqrt{(R+h_k)^2 - R^2\cos^2 E_o}}{R+h_k}$$

and so

$$(3.12) \qquad \varphi_m = \cos^{-1} \frac{R\cos^2 E_o + \sin E_o \sqrt{(R+h_k)^2 - F^2\cos^2 E_o}}{R\cos^2 E_k + \sin E_k \sqrt{(R+h_k)^2 - R^2\cos^2 E_k}}.$$

(4.

§ 4. THE INCLUSION OF DOPPLER DATA

A single Doppler pass for a low-altitude satellite gives, essentially, a measurement only of the three quantities: Minimum slant-range S_k, time t_k of minimum slant-range, and speed v_k.

It is shown in reference [2] that if frequency f is measured with standard deviation σ_f and if there is negligible drift in transmitted frequency during a single pass, but if the actual transmitted f is treated as an unknown constant, the information matrix relative to the parameters S_k, t_k, v_k is

$$
(4.1)\qquad M^{(D)}_{\alpha\beta} = \left\| \begin{matrix}
\frac{v^2}{R^2c^2}\left(\frac{f}{\sigma_f}\right)^2 \frac{n}{4\rho}\left(\psi - \frac{1}{4}\sin 4\psi\right) & 0 & \frac{v}{Rc^2}\left(\frac{f}{\sigma_f}\right)^2 \frac{n}{2}\left(-\frac{5\psi}{2} + \sin 2\psi + \frac{1}{8}\sin 4\psi\right) \\
0 & \frac{v^4}{R^2c^2}\left(\frac{f}{\sigma_f}\right)^2 \frac{n}{4\rho}\left(3\psi - 2\sin 2\psi + \frac{1}{4}\sin 4\psi\right) & 0 \\
\frac{v}{Rc^2}\left(\frac{f}{\sigma_f}\right)^2 \frac{n}{2}\left(-\frac{5\psi}{2} + \sin 2\psi + \frac{1}{8}\sin 4\psi\right) & 0 & \frac{1}{c^2}\left(\frac{f}{\sigma_f}\right)^2 n\rho\left(2\tan\psi + \frac{1}{4}\psi - \sin 2\psi - \frac{1}{16}\sin 4\psi\right)
\end{matrix} \right\|
$$

To convert this into an information matrix relative to our seven parameters λ_α we denote the maximum elevation during pass k by E_k, and observe that:

$$
(4.2)\qquad \delta S_k = \delta z \sin E_k \pm \delta y \cos E_k \, .
$$

according as the station lies to the $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}$ of the orbit.

Next we introduce the notation:

$$\left(\frac{a_N}{\mu}\right)^{1/2} \frac{\partial v}{\partial \lambda_\alpha} = v_\alpha \,, \tag{4.3}$$

so that, since

$$v \cong r\dot{\theta} = \frac{\sqrt{\mu \ell}}{r} \cong \left(\frac{\mu}{a}\right)^{1/2} \left[1 + \epsilon \cos\,(\theta - \beta)\right]$$

$$\cong \left(\frac{\mu}{a_N}\right)^{1/2} \left[1 - \frac{1}{2}\lambda_1 + \lambda_2 \cos\theta + \lambda_3 \sin\theta + \frac{1}{2}\lambda_5 - \lambda_5 \frac{I_1(\xi_N)}{I_o(\xi_N)} \cos\,(\theta - \beta_N)\right] ,$$

we have:

$$\begin{cases} v_1 = -\dfrac{1}{2} \\ v_2 = \cos\theta \\ v_3 = \sin\theta \\ v_4 = 0 \\ v_5 = \theta\left[\dfrac{1}{2} - \dfrac{I_1(\xi_N)}{I_o(\xi_N)} \cos\,(\theta - \beta_N)\right] \\ v_6 = v_7 = 0 \end{cases} \tag{4.4}$$

Making use of (4.2), (4.9), and the notation of (2.5), and replacing a_N by R, the matrix $M^{(D)}_{\lambda\mu}$ is convertible into the 7x7 matrix:

$$
\begin{aligned}
M_{\alpha\beta} = R^2 M^{(D)}_{11}\left(\pm y_\alpha \cos E_k + z_\alpha \sin E_k\right)^2 \cdot \left(\pm y_\beta \cos E_k + z_\beta \sin E_k\right) \\
+ 2R\left(\frac{\mu}{R}\right)^{1/2} M^{(D)}_{13}\left[v_\alpha\left(\pm y_\beta \cos E_k + z_\beta \sin E_k\right)\right. \\
\left. + v_\beta\left(\pm y_\alpha \cos E_k + z_\alpha \sin E_k\right)\right] \\
+ \frac{\mu}{R} M^{(D)}_{33}\, v_\alpha\, v_\beta + \frac{R^3}{\mu} M^{(D)}_{22}\, x_\alpha\, x_\beta
\end{aligned}
$$

Replacing v by $\sqrt{\frac{\mu}{R}}$, this is:

$$
\begin{aligned}
(4.5)\quad M_{\alpha\beta} = \frac{\mu}{c^2 R}\left(\frac{f}{\sigma_f}\right)^2 \Big\{ A'_k\, x_\alpha\, x_\beta + B'_k\left(\pm y_\alpha \cos E_k + z_\alpha \sin E_k\right)^2 \cdot \left(\pm y_\beta \cos E_k + z_\beta \sin E_k\right) \\
+ L_k\, v_\alpha\, v_k + P_k\left[v_\alpha\left(\pm y_\beta \cos E_k + z_\beta \sin E_k\right)\right. \\
\left. + v_\beta\left(\pm y_\alpha \cos E_k + z_\alpha \sin E_k\right)\right]\Big\}
\end{aligned}
$$

where

$$
(4.6)\quad
\begin{cases}
A'_k = \frac{n}{4\rho}\left(3\psi - 2\sin 2\psi + \frac{1}{4}\sin 4\psi\right) \\
B'_k = \frac{n}{4\rho}\left(\psi - \frac{1}{4}\sin 4\psi\right) \\
L_k = n\rho\left(2\tan\psi + \frac{1}{4}\psi - \sin 2\psi - \frac{1}{16}\sin 4\psi\right) \\
P_k = -n\left(\frac{5}{4}\psi - \frac{1}{2}\sin 2\psi - \frac{1}{16}\sin 4\psi\right)
\end{cases}
$$

§ 5. THE EFFECT OF RANDOM FLUCTUATIONS IN THE DRAG

Suppose that random fluctuations in the air drag have the effect of perturbing the time of arrival at a given orbital position without significantly perturbing the altitude.

A measurement $\tilde{q}_\ell$ is now related to a set of parameters λ_α by:

$$\tilde{q}_\ell = q(\lambda, t_\ell) + \frac{\partial q(t_\ell)}{\delta x}\, \delta x(t_\ell) + \sigma_\ell\, n_\ell \;, \tag{5.1}$$

where n_ℓ is a standard normal variable, and where $\delta x(t_\ell)$ is the perturbed position along the orbit at time t_ℓ . The $\delta x(t_\ell)$'s corresponding to different times t_ℓ are correlated through their dependence on previous air drag fluctuations (see below).

The least-squares fitting procedure described in § 1, which tacitly ignores any drag fluctuations, leads (as we have seen) to

$$\hat{\lambda}_\alpha - \lambda^o_\alpha = \sum_\beta (M^{-1})_{\alpha\beta} \sum_\ell \frac{1}{\sigma^2_\ell} \frac{\partial q(\lambda^o, t_\ell)}{\partial \lambda^o_\beta} \left[\tilde{q}_\ell - q_\ell(\lambda^o, t_\ell)\right] . \tag{5.2}$$

Linearization of (5.1) in the neighborhood of $\lambda = \lambda^o$ and substitution into (5.2) leads to:

$$\hat{\lambda}_\alpha - \lambda^o_\alpha = \sum_\beta (M^{-1})_{\alpha\beta} \sum_\ell \frac{1}{\sigma^2_\ell} \frac{\partial q(\lambda^o, t_\ell)}{\partial \lambda^o_\beta} \left[\sum_\gamma \frac{\partial q(\lambda^o, t_\ell)}{\partial \lambda^o_\beta} \left(\lambda_\gamma - \lambda^o_\gamma\right) + \frac{\partial q(t_\ell)}{\partial x}\, \delta x(t_\ell) + \sigma_\ell\, n_\ell\right] \tag{5.3}$$

But $\sum_{\imath} \frac{1}{\sigma_{\imath}^2} \frac{\partial q(\lambda^o, t_{\imath})}{\partial \lambda_{\beta}^o} \frac{\partial q(\lambda^o, t_{\imath})}{\partial \lambda_{\beta}^o} = M_{\beta\gamma}$. Equation (5.3) thus simplifies

to

$$(5.4) \quad \hat{\lambda}_{\alpha} - \lambda_{\alpha} = \sum_{\beta} (M^{-1})_{\alpha\beta} \sum_{\imath} \frac{1}{\sigma_{\imath}} \frac{\partial q(\lambda^o, t_{\imath})}{\partial \lambda_{\beta}^o} \left[n_{\imath} + \frac{1}{\sigma_{\imath}} \frac{\partial q(t_{\imath})}{\partial x} \delta x(t_{\imath}) \right] .$$

Denoting the covariance of the quantities $\delta x(t_{\imath})$. $\delta x(t_j)$ by $\mu_{\imath j}$ and assuming that these quantities are normally distributed with mean zero and, of course, independently of the measurement errors $\sigma_{\imath} n_{\imath}$, it follows that the covariance of the estimates λ_{α} is:

$$\mathcal{E} \left\{ (\lambda_{\alpha} - \lambda_{\alpha})(\lambda_{\beta} - \lambda_{\beta}) \right\} = \sum_{\gamma} \sum_{\delta} (M^{-1})_{\alpha\gamma} (M^{-1})_{\beta\delta} \left\{ \sum_{\imath} \frac{1}{\sigma_{\imath}^2} \frac{\partial q(\lambda^o, t_{\imath})}{\partial \lambda_{\gamma}^o} \frac{\partial q(\lambda^o, t_{\imath})}{\partial \lambda_{\delta}^o} \right.$$

$$\left. + \sum_{\imath} \sum_{j} \frac{1}{\sigma_{\imath}^2} \frac{1}{\sigma_j^2} \mu_{\imath j} \frac{\partial q(\lambda^o, t_{\imath})}{\partial \lambda_{\gamma}^o} \frac{\partial q(\lambda^o, t_j)}{\partial \lambda_{\delta}^o} \frac{\partial q(t_{\imath})}{\partial x} \frac{\partial q(t_j)}{\partial x} \right\}$$

This simplifies to

$$(5.5) \qquad \mathcal{E} \left\{ (\lambda_{\alpha} - \lambda_{\alpha})(\lambda_{\beta} - \lambda_{\beta}) \right\} = (M^{-1})_{\alpha\beta} + (M N M^{-1})_{\alpha\beta}$$

where

$$(5.6) \quad N_{\alpha\beta} = \sum_{\imath} \sum_{j} \mu_{\imath j} \left(\frac{1}{\sigma_{\imath}^2} \frac{\partial q(\lambda^o, t_{\imath})}{\partial \lambda_{\alpha}^o} \frac{\partial q(t_{\imath})}{\partial x} \right) \left(\frac{1}{\sigma_j^2} \frac{\partial q(\lambda^o, t_j)}{\partial \lambda_{\beta}^o} \frac{\partial q(t_j)}{\partial x} \right) .$$

Next, as in § 3, we group the summation over observations into summations over passes. Like $\frac{\partial \vec{r}}{\partial \lambda}$, $\mu_{\imath j}$ may be regarded as effectively constant when $\imath$ runs over observations during a single pass and j runs over observations during another pass, at least if the correlation time of the

$\delta x(t)$'s is fairly long, as we may expect. Using the particular form of (4.5) and the matrices (3.4) and (3.8) wherein x is not coupled to y , z , the contribution of a single pass k to the sum

$$\sum_{\ell} \frac{1}{\sigma_{\ell}^{2}} \frac{\partial q(\lambda^{o}, t_{\ell})}{\partial \lambda_{\alpha}^{o}} \frac{\partial q(t_{\ell})}{\partial x}$$

is just

$$\frac{1}{a_N^2} A_k \frac{\partial x(t_k)}{\partial \lambda_\alpha^o} = \frac{1}{a_N} A_k x_\alpha$$

where

(5.7) $$A_k = \left(\frac{a_N}{\sigma_S}\right)^2 (Q_S)_{11} + \left(\frac{a_N}{R\sigma_E}\right)^2 (Q_E)_{11} + \frac{\mu}{c^2 R} \left(\frac{f}{\sigma_f}\right)^2 A'_k$$

Equation (5.6) thus simplifies to

(5.8) $$N_{\alpha\beta} = \sum_k \sum_\ell \frac{\mu_{k\ell}}{a_N^2} A_k A_\ell \; x_\alpha x_\beta$$

where the summations $\sum_k$ and $\sum_\ell$ are now summations over different passes.

Before evaluating the covariances $\mu_{k\ell}$ of the $\delta x(t_k)$'s we must note that the true position is:

(5.9) $$x(t) = x(\lambda, t) + \delta x(t)$$

where λ are true parameters at some time t_o , and $\delta x(t)$ is the accumulated position error since t_o , expressible in the form (cf. § 6):

(5.10) $$\delta x(t) = \int_{t_o}^{t} (t - t') \, \delta\ddot{x}(t') \, dt' .$$

The error in position determination along the orbit at time t^* is:

$$\epsilon_x(t^*) = x(\hat{\lambda}, t^*) - x(\lambda, t^*) - \delta x(t^*) \tag{5.11}$$

Choosing t_o to coincide with t^*, we have $\delta x(t^*) = 0$ and

$$\sigma^2_{x(t^*)} = \sum_\alpha \sum_\beta \left[(M^{-1})_{\alpha\beta} + (M^{-1} N M^{-1})_{\alpha\beta} \right] \frac{\partial x(\lambda, t^*)}{\partial \lambda_\alpha} \frac{\partial x(\lambda, t^*)}{\partial \lambda_\beta} \tag{5.12}$$

The first term in the [ ] reproduces the variance due to measurement errors described in § 1. The second term yields the contribution of drag fluctuations.

§ 6. EVALUATION OF THE $\mu_{k\ell}$'s

It is easy to show from energy considerations that a small change in velocity Δv along a near-circular orbit yields a change in semi-major axis given by:

$$\Delta\lambda_1 = \frac{\Delta a}{a} = 2 \frac{\Delta v}{v} \tag{6.1}$$

If the change occurs at angular position θ' from the equator, since r and $\frac{1}{r}\frac{dr}{d\theta}$ must remain unchanged at this position we have, in the notation of § 2:

$$\begin{aligned} \Delta\lambda_2 \cos\theta' + \Delta\lambda_3 \sin\theta' &= \Delta\lambda_1 \\ -\Delta\lambda_2 \sin\theta' + \Delta\lambda_3 \cos\theta' &= 0 \end{aligned} \tag{6.2}$$

so that

$$(6.3) \qquad \Delta\lambda_2 = 2\,\frac{\Delta v}{v}\cos\theta' \qquad \Delta\lambda_3 = 2\,\frac{\Delta v}{v}\sin\theta'$$

Substitution into (2.4) yields a change in subsequent distance along the orbit, given by:

$$(6.4) \qquad \frac{\Delta x(\theta)}{a_N} = \frac{\Delta v}{v}\left[-3(\theta-\theta') + 4\sin(\theta-\theta')\right]$$

Now the retardation due to drag is given by

$$(6.5) \qquad \frac{dv}{dt} = -B\rho v^2 \quad ,$$

where ρ is air-density and $B = \frac{1}{2}C_D A/m$, C_D being the satellite drag-coefficient, A the reference area, and m the satellite mass.

The effect on position $x(\theta)$ due to drag between θ^* and θ is thus expressible by:

$$\frac{\Delta x(\theta)}{a_N} = \int_{\theta'=\theta^*}^{\theta} B\rho v\left[3(\theta-\theta') - 4\sin(\theta-\theta')\right]dt' \quad ,$$

where $\frac{d\theta'}{dt'} = \frac{v}{a_N}$, so that

$$(6.6) \qquad \frac{\Delta x(\theta)}{a_N} = \int_{\theta^*}^{\theta}(B\rho a_N)\left[3(\theta-\theta') - 4\sin(\theta-\theta')\right]d\theta' \quad .$$

The effect of an exponentially decreasing density-altitude relationship is accounted for by the parameter λ_5 of § 2. The effect of a deviation $\delta\rho(\theta')$ in density, at angular position θ', from that given by the exponential formula yields a change in position $\delta x(\theta)$ from that corresponding to the set λ of parameters which fit perfectly at θ^*, namely:

$$(6.7) \qquad \delta x(\theta) = a_N \int_{\theta^*}^{\theta} B\, a_N\, \delta\rho(\theta') \left[3\,(\theta - \theta') - 4 \sin\,(\theta - \theta') \right] d\theta' \,.$$

Next, we shall assume that $\delta\rho(\theta')$ is a stationary time-series with negative exponential auto-correlation, so that

$$(6.8) \qquad \mathcal{E}\left\{ \delta\rho(\theta_1)\ \delta\rho(\theta_2) \right\} = \sigma_\rho^2\, e^{-|\theta_1 - \theta_2|/\theta_c}$$

where σ_ρ^2 is the variance of $\delta\rho$ and θ_c the non-dimensional "correlation time" of the time series.

Assuming that $\theta_c > 2\pi$, say, and that the sum (5.8) contains passes over several revolutions, the second term in the square bracket inside the integral (6.7) is unimportant by comparison with the first term. Omitting the unimportant term and using (6.8) we obtain:

$$(6.9) \qquad \mu_{k\ell} = \mathcal{E}\left\{ \delta x(\theta_k)\ \delta x(\theta_\ell) \right\} \cong a_N^2 (3\, B\, a_N\, \sigma_\rho)^2 \int_{\theta^*}^{\theta_k} \int_{\theta^*}^{\theta_\ell} (\theta_k - \theta_1)(\theta_\ell - \theta_2)\, e^{-|\theta_1 - \theta_2|/\theta_c}\, d\theta_1\, d\theta_2 \,.$$

Finally, the double integral in (6.9) may be evaluated to yield:

$$(6.10) \qquad \mu_{k\ell} = \frac{3}{2}\, a_N^2\, (B\, a_N\, \sigma_\rho)^2 \Big\{ f(|\theta_k - \theta_\ell|) - f(|\theta^* - \theta_k|) - f(|\theta^* - \theta_\ell|) \pm 3(\theta^* - \theta_\ell)\, f(|\theta^* - \theta_k|) \pm 3(\theta^* - \theta_k)\, f(|\theta^* - \theta_\ell|) \Big\}$$

where the ambiguities in sign correspond to $\theta^* \gtrless \theta_k$ and $\theta^* \gtrless \theta_\ell$ respectively and where:

$$
(6.11) \quad \begin{cases} f(\theta) = \displaystyle\int_0^\theta \theta^3 e^{-\theta/\theta_c} \, d\theta = \theta_c \Big\{ \theta^3 - 3\,\theta^2\,\theta_c + 6\,\theta\,\theta_c^2 \\ \qquad\qquad - 6\,\theta_c^3 + 6\,\theta_c^3\, e^{-\theta/\theta_c} \Big\} , \\ g(\theta) = \displaystyle\int_0^\theta \theta^2 e^{-\theta/\theta_c} \, d\theta = \theta_c \Big\{ \theta^2 - 2\,\theta\,\theta_c^2 + 2\,\theta_c^2 \\ \qquad\qquad - 2\,\theta_c^2\, e^{-\theta/\theta_c} \Big\} . \end{cases}
$$

REFERENCES

[1] John V. Breakwell; "The Effect of Air Drag on Elliptical Orbits Around a Non-Rotating Spherical Earth," Internal Memorandum, October 1957.

[2] John V. Breakwell; "The Local Accuracy of Doppler Orbit Determination," LMSD-285536, October 9, 1959.